Counting Fibonacci Identities

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Fibonacci and Lucas Numbers

We define the Fibonacci sequence as follows:

$$f_0 = 1, \quad f_1 = 1,$$
$$f_n = f_{n-1} + f_{n-2}.$$

We define the Lucas sequence similarly:

$$L_0 = 2, \quad L_1 = 1,$$
$$L_n = L_{n-1} + L_{n-2}.$$

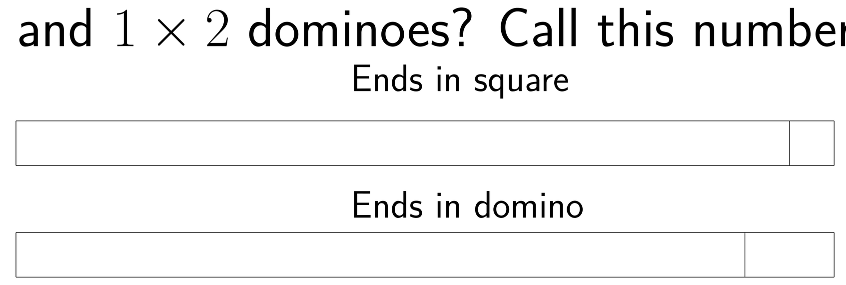
Zeckendorf's Theorem

Every natural number $n \in \mathbb{N}$ can be represented uniquely a series of non-consequitive Fibonacci numbers, excluding f_0. We call such a series the Zeckendorf Representation(ZR) of n. For example:

$$10 = 2 + 8 = f_2 + f_6.$$

Combinatorial Interpretation of f_n

How many ways can you tile a $1 \times n$ board with 1×1 squares and 1×2 dominoes? Call this number f_n.



There are f_{n-1} ways to tile an n-board ending in a square and f_{n-2} ways to tile an n-board ending in a domino.

Combinatorial Interpretation of L_n

How many ways can you tile a $1 \times n$ circular board with 1×1 curved squares and 1×2 curved dominoes? Call this number L_n.

A similar argument holds with circular boards as straight boards. Further, we can break every circular board into an n-board and $(n-2)$-board, those which are in phase and those which are out of phase. Thus

$$L_n = f_n + f_{n-2}$$

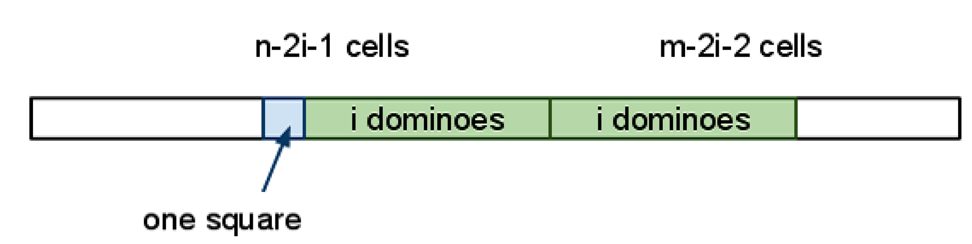
Zeckendorf Representation of $L_m L_n$

The following is the statement of the ZR of $L_{2k}L_n$. For integers k and n such that $n - 2 > 2k > 1$,

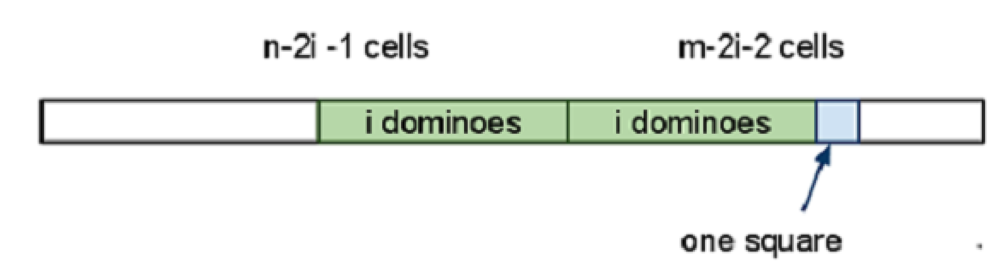
$$L_{2k}L_n = f_{n+2k} + f_{n+2k-2} + f_{n-2k} + f_{n-2k-2}.$$

In order to prove this theorem, we ask the question, how many can a $n + m - 2$ board have a fault at cell n but not at $n - 1$. Let A be the set of all tilings of an $(n + m - 2)$-board witha fault at $n - 2$. Let B be the set of all tilingso of an $n + m - 2$ board with a fault at $n - 1$. We will break A and B into many disjoint sets indexed by the the square closest to the fault.

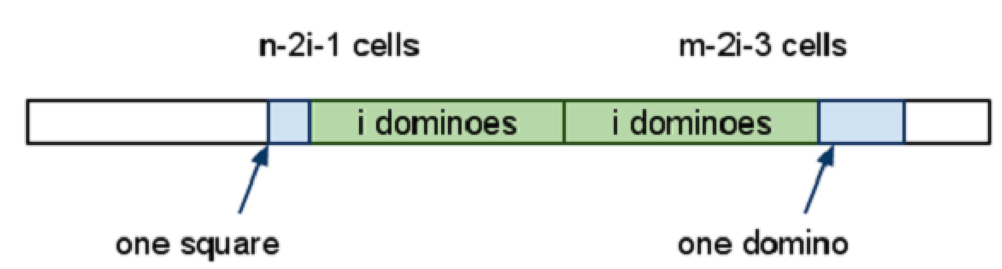
The following are tilings in A with the nearest sqaure i domonies from the fault and on the left or $A_2 i$.



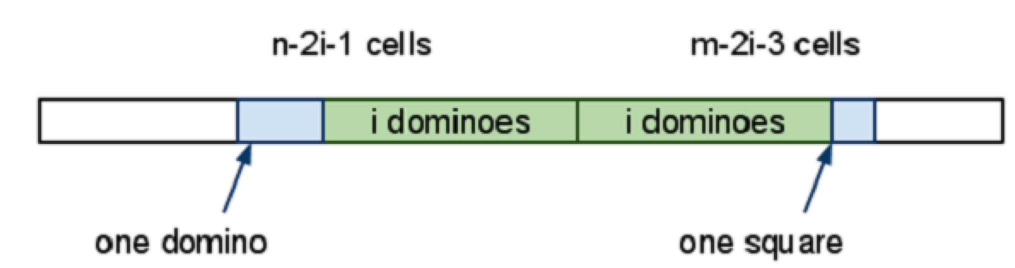
The following are tilings in B with the nearest sqaure i domonies from the fault and on the right or $B_2 i$.



The following are tilings in A with nearest square i dominoes from the fault and on the right side.



The following are tilings in B with nearest square i dominoes from the fault and on the left side.



Clearly most of the elements of A and B cancel. We now only care what the final remainder after our subtraction is. In the case when m is even A contains a $n - m$ board, and when m is odd B contains an $(n - m)$-board. Thus

$$f_{n-2}f_m - f_{n-1}f_{m-1} = (-1)^m f_{n-m}$$

Careful application of this lemma led to the ZR of $L_m L_n$. Further, we can generate many triple product ZR's by applying the even case of the lemma and placing known ZR in the middle, bellow are some interesting results.

New Zeckendorf Representations

For $n > 2j > m$ and $n > 2j + m$,

$$f_m L_{2j} f_n = \begin{cases} f_{n+2j-m} + f_{n-2j-m} + \Sigma_{i=1}^{\frac{m-2}{2}} f_{n+2j+m-4i-1} + \Sigma_{i=1}^{\frac{m-2}{2}} f_{n-2j+m-4i-1} & \text{for } m \text{ even,} \\ \Sigma_{i=1}^{\frac{m+1}{2}} f_{n+2j-m-3+4i} + \Sigma_{i=1}^{\frac{m+1}{2}} f_{n-2j-m-3+4i} & \text{for } m \text{ odd.} \end{cases}$$

For $n > m > 2j$ and $n > m + 2j$,

$$L_{2j} f_m f_n = \begin{cases} f_{n-m+2j-1} + f_{n+m-2j} + \sum_{i=1}^{j} f_{n-m-2j-3+4i} & \\ \quad + \sum_{i=1}^{j} f_{n+m-2j-1+4i} + \sum_{i=1}^{\frac{m}{2}-j} f_{n-m+2j+4i} & \text{for } m \text{ odd,} \\ f_{n-m-2j} + f_{n+m-2j} + \sum_{i=1}^{j} f_{n-m-2j-1+4i} & \\ \quad + \sum_{i=1}^{j} f_{n+m-2j-1+4i} + \sum_{i=1}^{\frac{m}{2}-j} f_{n-m+2j-2+4i} & \text{for } m \text{ even.} \end{cases}$$

Future Work

The primary lemma seems to hold the key to many interesting Zeckendorf Representations involving Lucas numbers We did, however, have little luck finding closed form Zeckendorf representation of $f_p L_m f_n$ where m is odd.

Our new Zeckendorf Representations are proven using many combinatorial mappings of our boards and bracelets to produce their Zeckendorf Representations. We believe much insight into the problem could be found by proving each with a single mapping.

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References

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